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LOCALIZATION FUNCTOR $S^{-1}()$ AND FUNCTORS EXT AND TOR

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Abstract. In this paper, unless otherwise stated, B is a duo-ring, A a sub-ring of B, S a multiplicatively closed subset of the ring A satisfying the left (right respectively) Ore's conditions, $\overline{S}$ the set of regular elements of S, ${}_AM$ a left $A-$module, ${}_BN_A$ a $(B-A)-$bimodule (respectively ${}_AN_B$ a $(A-B)-$bimodule), $S^{-1}A$ the ring of fractions of A on S; particularly, if A is a duo-ring and P is a prime ideal of A then S is the set of regular elements of $A-P$.

We establish the following results:

1) Isomorphisms of left $(\overline{S})^{-1}B-$modules:
$$(\overline{S})^{-1}Hom_A(M,N) \cong Hom_{S^{-1}A}(S^{-1}M, S^{-1}N))$$
where M is an $(A-B)-$bimodule and N a left $A-$module;
2) Isomorphisms of left $(\overline{S})^{-1}B-$modules:
$$(\overline{S})^{-1}Tor_n^A(M,N) \cong Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N) \quad \text{and}$$
$$Tor_n^A(M,N)_P \cong Tor_n^{A_P}(M_P, N_P);$$
3) Isomorphisms of right $(\overline{S})^{-1}B-$modules:
$$(\overline{S})^{-1}Ext_A^n(M,N) \cong Ext_{S^{-1}A}^n\left(S^{-1}M, S^{-1}N\right) \quad \text{and}$$
$$Ext_A^n(M,N)_P \cong Ext_{A_P}^n(M_P, N_P);$$
4) If B is noetherian, ${}_BM_A$ a $(B-A)-$bimodule, then the functors:
$$Ext^n_{(\overline{S})^{-1}B}(S^{-1}M,-) : (\overline{S})^{-1}B - Mod \to S^{-1}A - Mod \quad \text{and}$$
$Tor_n^{S^{-1}A}(S^{-1}M,-) : S^{-1}A - Mod \to (\overline{S})^{-1}B - Mod$ are adjoint.

Key words and phrases. Ring, duo-ring, left (right) Ore's conditions, multiplicatively closed subset, left $A-$module, ring of fractions, module of fractions, category $A-$Mod, Mod$-A$, functors $S^{-1}()$, Ext and Tor.

Introduction

The aim of this paper is to study in the case where the ring A is not commutative some fundamental homological algebra results in the category of left A -modules, $A-$Mod (respectively right $A-$modules Mod$-A$) which are true in the case where A is commutative.
The transition from the commutative to the not necessarily commutative is not obvious. For example, if A is commutative, M and N are two left A -modules, $Hom_A(M,N)$, $Ext_A^n(M,N)$ and $Tor_n^A(M,N)$ are left $A-$modules, which is not generally true in the case where A is not commutative (see [1] and [9]). Also, some relations between the functor $S^{-1}()$ and the homological functors Ext and Tor are established by several authors in the case the ring A is commutative (see [3], chap IV and [9], chap $VIII$ and XI).
So, to study the localization and the functorization in the case where A is not commutative becomes important (see [2]).
Let us recall for this some useful cases for this work:
Let A, B and C be three rings.
-If M is an $(A-B)-$bimodule and N a left $A-$module, then $Hom_A(M,N)$ and $Ext_n^A(M,N)$ are left $B-$modules.
-If M is a left $A-$module and N a $(A-B)-$bimodule, then $Hom_A(M,N)$ and $Ext_n^A(M,N)$ are right $B-$modules.
-If M is a $(B-A)-$bimodule and N a $(A-C)-$bimodule, then $M\otimes_A N$ and $Tor_A^n(M,N)$ are $(B-C)-$bimodules.
-If M is a $(A-B)-$bimodule and N a left $B-$module, then $M\otimes_A N$ and $Tor_A^n(M,N)$ are left $A-$modules.
-If M is a left A- module and N is $(B-A)-$bimodule, then $M\otimes_A N$ and $Tor_A^n(M,N)$ are left $B-$modules.
So in this paper, unless otherwise stated, B is a unitary duo-ring, A a subring of B, S a multiplicatively closed subset of the ring A satisfying the left(right respectively) Ore's conditions, $\overline{S}$ the set of regular elements of S, M and N are left $A-$modules (respectively right), $S^{-1}A$, the ring of fractions of A on S and $S^{-1}M$ the left $S^{-1}A-$module of fractions of M on S (right respectively), n a integer.
Using the works of Mohamed Ben Fraj Ben MAAOUIA, especially in his third- cycle thesis ([4]), his state thesis ([5]), his papers with Mr. Mamadou SANGHARE: "Rings and Modules of fractions" ([6]) and "Localization in duo-rings" [7]), about localization and the functor $S^{-1}() : A-Mod \longrightarrow S^{-1}(A)-Mod$ which is isomorphic to functor $S^{-1}A\ \otimes_A - : A-Mod \longrightarrow S^{-1}(A)-Mod$ and which conserves projectivity, we have established the following results:
In the first section, we have proved the isomorphisms of right $(\overline{S})^{-1}B-$modules:

$$(\overline{S})^{-1}Hom_A(M,N) \cong Hom_{S^{-1}A}(S^{-1}M, S^{-1}N))$$

and

$$Hom_{S^{-1}A}(S^{-1}N\otimes S^{-1}M, S^{-1}Q) \cong Hom_{S^{-1}A}(S^{-1}M, Hom_{S^{-1}A}(S^{-1}N, S^{-1}Q).$$

In the second section, we have proved the isomorphisms of left $(\overline{S})^{-1}B-$modules:

$$(\overline{S})^{-1}Tor_n^A(M,N) \cong Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N)$$

and $Tor_n^A(M,N)_P \cong Tor_n^{A_P}(M_P, N_P)$ where A is a duo-ring and P is a prime ideal of A and $M_P = S^{-1}M$ is the fractions module of M where S is the set of

regular elements of $A-P$.
In the third section: If A is noetherian and M is finite type module, then:

$$(\overline{S})^{-1}Ext_A^n(M,N) \cong Ext_{S^{-1}A}^n\left(S^{-1}M, S^{-1}N\right)$$

and $Ext_A^n(M,N)_P \cong Ext_{A_P}^n(M_P,N_P)$ where A is a duo-ring and P is a prime ideal of A.
In the fourth section, we are interested in the adjoint notion between the functor $S^{-1}()$ and the homological functors $Ext_A^n(M,-)$ and $Tor_n^A(M,-)$ Which are not studied according to us.
In the case where the ring is not necessarily commutative, M. Maaouia established in his works that the functor $S^{-1}()$ is adjoint to the functor $Hom_A(S^{-1}A-)$ (see [5], chap I and [6]).
We know that (see [10]) the functor $Ext_A^0(M,-)$ is equivalent to the functor $Hom_A(M,-)$, and $Tor_0^A(M,-)$ is equivalent to the functor $M\otimes_A -$ Which is isomorphic to the functor $S^{-1}()$.
So $Ext_{S^{-1}A}^0(S^{-1}A,-)$ is adjoint to $S^{-1}()$ and $Tor_0^{S^{-1}A}(S^{-1}A,-)$ is isomorphic to $S^{-1}()$. We generalize this result for any integer n as following.
If B denotes a unitary duo-ring, A a subring of B, M a left finite type $A-$module, then $Ext_{(\overline{S})^{-1}B}^n(S^{-1}M,-)$ and $Tor_n^{S^{-1}A}(S^{-1}M,-)$ are adjoint functors for any integer n.

1. Basic definitions and Preliminary results

1.1. Basic definitions.

Definition 1.1. *Let A be a ring and S a non empty subset of A non containing zero. Then, we say that S is a multiplicatively closed subset of the ring A if :*

i) $\forall s,t\in S$, $st\in S$ *(we say that S is stable for multiplication)*
ii) $\forall s,t\in A$ *such as* $st\in S$, *then* $s\in S$ *and* $t\in S$*(we say that S is closed)*

Definition 1.2. *If S is a multiplicatively closed subset of the ring A, we say that S satisfies the left Ore's conditions (respectively right) if :*

i) $\forall a\in A, \forall s\in S$, *there are* $t\in S$ *and* $b\in A$ *such as* $ta=bs$ *(respectively* $at=sb$*). We say that S is left switchable (respectively right).*
ii) $\forall a\in A, \forall s\in S$ *such as* $as=0$ *(respectively* $sa=0$*), then there is* $t\in S$ *such as* $ta=0$ *(respectively* $at=0$*). We say that S is left reversible (respectively right).*

Definition 1.3. *A ring A is a duo-ring if each ideal of A is a two-sided ideal.*

Proposition 1.4. *A ring A is a duo-ring if and only if for every element a of A, we have $aA=Aa$.*

Definition 1.5. *Let A and B be two rings. We say that M is an $(A-B)-$bimodule denoted ${}_AM_B$ if M is a left $A-$module and a right $B-$module.*

Proposition 1.6. *Let B be a duo-ring, A a subring of B, S a non-empty multiplicatively closed subset of the ring A satisfying the left (respectively right) Ore's conditions. Then $\overline{S}$ the set of regular elements of S satisfies the left Ore's conditions in B (respectively right).*

Proof. S is a non-empty multiplicatively closed subset of the ring A then it is a non-empty multiplicatively closed subset of the ring B. then $\overline{S}$, the set of regular elements of S is a multiplicatively closed subset of the ring B satisfying the left Ore's conditions (respectively right) (see [5], chap II).

□

Proposition 1.7. *Let M be a left (respectively right) $A-$module and S a multiplicatively closed subset of the ring A satisfying the left (respectively right) Ore's conditions. Then the binary relation R defined in $S \times M$ (respectively $M \times S$) by:*
$(s,m)R(s',m')$ if and only if there is $x, y \in S$ such as $\begin{cases} xm = ym' \\ xs = ys' \end{cases}$
(respectively $(m,s)R(m',s')$ if and only if there is $x', y' \in S$ such as $\begin{cases} mx' = m'y' \\ sx' = s'y' \end{cases}$)
is an equivalence relation(see [5] and [6]).

Proof. See [5]. □

Notation 1.8. *The set $(S\times M)/R$ (respectively $(M\times S)/R$) of equivalence classes is denoted $S^{-1}M$, which elements are denoted $\frac{m}{s}$ where $m \in M$ and $s \in S$.*
If $M = A$, the set $(S \times A)/R$ is denoted $S^{-1}A$ which elements are denoted $\frac{a}{s}$ where $a \in A$ and $s \in S$.

Theorem 1.9. *Let M be a left (respectively right) $A-$module and S a multiplicatively closed subset of the ring A satisfying the left (respectively right) Ore's conditions, then:*

1) *$S^{-1}A$ has a ring structure defined by the operations:*
   - $\frac{a}{t} + \frac{b}{s} = \frac{xa + yb}{ys}$ *where $x, y \in S$ such as $xt = ys$ and*
   - $\frac{a}{t} \times \frac{b}{s} = \frac{zb}{wt}$ *where $(w, z) \in S \times A$ such as $wa = zs$*

   (respectively
   - $\frac{a}{t} + \frac{b}{s} = \frac{ax + by}{sy}$ *where $x, y \in S$ such as $tx = sy$ and*
   - $\frac{b}{s} \times \frac{a}{t} = \frac{bz}{tw}$ *where $(w, z) \in S \times A$ such as $aw = sz$),(see [5] and [6])*
2) *$S^{-1}M$ is a left $S^{-1}A-$module (respectively right $S^{-1}A-$module). Laws of $S^{-1}M$ are defined by:*
   - $\frac{m}{s} + \frac{m'}{s'} = \frac{xm + ym'}{ys'}$ *where $x, y \in S$ such as $xs = ys'$ and*
   - $\frac{a}{t}.\frac{m}{s} = \frac{zm}{wt}$ *where $(w, z) \in S \times A$ such as $wa = zs$.*

   (respectively
   - $\frac{m}{s} + \frac{m'}{s'} = \frac{mx + m'y}{s'y}$ *where $x, y \in S$ such as $sx = s'y$ and*
   - $\frac{m}{s}.\frac{a}{t} = \frac{mz}{tw}$ *where $(w, z) \in S \times A$ such as $aw = sz$)(see [5] and [6]).*
3) *The relation*
   $S^{-1}() : A - Mod \longrightarrow S^{-1}A - Mod$ (respectively $S^{-1}() : Mod - A \longrightarrow Mod - S^{-1}A$) such as for any $M \in A - Mod$ (respectively $M \in Mod - A$)associates $S^{-1}M$ and for any morphism of left $A-$module (respectively

right $A-$module) $f : M \longrightarrow N$ associates the morphism of left (respectively right) $S^{-1}A-$modules
$S^{-1}f : S^{-1}M \to S^{-1}N$
$$\frac{m}{s} \mapsto \frac{f(m)}{s}$$
is covariant functor witch is additive, exact and preserves projectivity (see [5] and [6]). In particularity, this functor preserves injectivity if A is noetherian (see [2]).

1.2. **Preliminary results.**

Theorem 1.10. *Let B be a duo-ring (not necessarily commutative), A a subring of B, S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions in A, $\overline{S}$ the set of regular elements of S satisfying the right Ore's conditions in B, M an $(A-B)-$bimodule. Then for every left $A-$module N, we have the following isomorphism of left $(\overline{S})^{-1}B-$modules:*
$\psi : (\overline{S})^{-1}Hom_A(M,N) \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N))$
$$\frac{f}{s} \mapsto \frac{1}{s}S^{-1}(f).$$

Proof. It's clear that $Hom_A(M,N)$ is a left $B-$module so $(\overline{S})^{-1}Hom_A(M,N)$ is a left $(\overline{S})^{-1}B-$module; $S^{-1}M$ is a $(S^{-1}A - (\overline{S})^{-1}B)-$bimodule, $S^{-1}N$ is a left $S^{-1}A-$module so $Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$ is a left $(\overline{S})^{-1}B-$module.

a) Let us show that the correspondence ψ is a map.
Let $\frac{f}{s}$ and $\frac{g}{t}$ be two elements of $\overline{S}^{-1}Hom_A(M,N)$, such as $\frac{f}{s} = \frac{g}{t}$.
Let us show that $\psi(\frac{f}{s}) = \psi(\frac{g}{t})$. We have $\frac{f}{s} = \frac{g}{t}$ and $\overline{S}$ satisfies the right Ore's conditions in B, then there are x and $y \in \overline{S}$ such as:
$\begin{cases} fx = gy \\ sx = ty \end{cases}$.
$f.x = g.y \Rightarrow \forall m \in M, (f.x)(m) = (g.y)(m) \Rightarrow f(m.x) = g(m.y)$
Let $\frac{m}{l} \in S^{-1}(M)$
Let us show that $\psi(\frac{f}{s}) = \psi(\frac{g}{t})$, that means $[\frac{1}{s}S^{-1}(f)](\frac{m}{l}) = [\frac{1}{t}S^{-1}(g)](\frac{m}{l})$ for every $\frac{m}{l} \in S^{-1}(M)$.
We have
$\psi(\frac{f}{s})(\frac{m}{l}) = \frac{1}{s}S^{-1}(f)(\frac{m}{l}) = S^{-1}(f)(\frac{m}{l}.\frac{1}{s}) = S^{-1}(f)(\frac{mz}{sw}) = \frac{f(mz)}{sw}$
where $(w,z) \in \overline{S} \times B$ such as $w.1 = z.l$;
$\psi(\frac{g}{t})(\frac{m}{l}) = \frac{1}{t}S^{-1}(g)(\frac{m}{l}) = S^{-1}(g)(\frac{m}{l}.\frac{1}{t}) = S^{-1}(g)(\frac{mz'}{tw'}) = \frac{g(mz')}{tw'}$
where $(w',z') \in \overline{S} \times B$ such as $w'.1 = z'.l$
Let us show that for every $\frac{m}{l} \in S^{-1}(M)$, there are $(w,z) \in \overline{S} \times B$ and $(w',z') \in \overline{S} \times B$ such as $\frac{f(mz)}{sw} = \frac{g(mz')}{tw'}$
For every $m \in M$, there are x and $y \in S$ such as:
$\begin{cases} f(m.x) = g(m.y) \\ sx = ty \end{cases}$ Then we can take $\begin{cases} z = x \\ z' = y \end{cases}$,

then we have $f(mz) = g(mz')$ and so $\begin{cases} sw = szl = sxl \\ tw' = tz'l = tyl \end{cases}$.

So $\begin{cases} f(m.z) = g(m.z') \\ sxl = tyl \end{cases} \Rightarrow \begin{cases} f(m.z).1 = g(m.z').1 \\ sxl = tyl \end{cases} \Rightarrow \frac{f(mz)}{sx} = \frac{g(mz')}{ty}$

Then taking $\begin{cases} w = x \\ w' = y \end{cases}$, we have proved that there are $(w, z) \in \overline{S} \times B$ and $(w', z') \in \overline{S} \times B$ such as $\frac{f(mz)}{sw} = \frac{g(mz')}{tw'}$.

Then $\psi(\frac{f}{s}) = \psi(\frac{g}{t})$, hence ψ is a map.

b) Let us now show that ψ is a morphism of left $(\overline{S})^{-1}B-$module.

We have $\psi(\frac{f}{s} + \frac{g}{t}) = \psi(\frac{f.x + g.y}{ty})$ where $sx = ty$

$\Longrightarrow \psi(\frac{f}{s} + \frac{g}{t}) = \frac{1}{ty} S^{-1}(f.x + g.y)$

For any morphism $h : M \to N$ of right $B-$module, the morphism $S^{-1}h : S^{-1}M \to S^{-1}N$ is $\overline{S}^{-1}B$ linear and the functor $S^{-1}()$ is additive. Then

$\frac{1}{ty} S^{-1}(f.x + g.y) = \frac{S^{-1}(f).x + S^{-1}(g).y}{ty} =$

$= \frac{1}{ty} S^{-1}(f).x + \frac{1}{ty} S^{-1}(g).y = \psi(\frac{f}{s}) + \psi(\frac{g}{t})$.

Therefore let us show that $\psi(\frac{f}{s}.\frac{a}{t}) = \frac{a}{t}.\psi(\frac{f}{s})$.

Let $\frac{m}{l} \in S^{-1}(M)$, we have

$\psi(\frac{f}{s}.\frac{a}{t})(\frac{m}{l}) = \psi(\frac{f.z}{tw})(\frac{m}{l})$ where $(w, z) \in (\overline{S}) \times B$ such as $aw = sz$. Then

$\psi(\frac{f}{s}.\frac{a}{t})(\frac{m}{l}) = \frac{1}{tw} S^{-1}(fz)(\frac{m}{l}) = S^{-1}(fz)(\frac{m}{l}.\frac{1}{tw}) = S^{-1}(fz)(\frac{mz'}{tw.w'}) = \frac{f(m.z'.z)}{(tw).w'}$ where $(w', z') \in (\overline{S}) \times B$ such as $w'.1 = z'.l$.

Otherwise $\frac{a}{t}.\psi(\frac{f}{s})(\frac{m}{l}) = \frac{a}{t}.\frac{f(mz_1)}{sw_1}$ where $w_1.1 = z_1.l$.

Then $\frac{a}{t}.\psi(\frac{f}{s})(\frac{m}{l}) = \frac{z_2 f(mz_1)}{w_2 t}$ where $w_2.a = z_2(sw_1)$.

Then $\frac{a}{t}.\psi(\frac{f}{s})(\frac{m}{l}) = \frac{f(mz_1z_2)}{w_2 t})$.

B is a duo-ring, then there is $w_3 \in B$ such as $w_2 t = tw_3$.

Then $\frac{a}{t}.\psi(\frac{f}{s})(\frac{m}{l}) = \frac{f(mz_1z_2)}{tw_3}$.

To show that $\frac{f(m.z'.z)}{(tw).w'} = \frac{f(mz_1z_2}{w_2 t})$ we can take $\begin{cases} z_1 = z' \\ z_2 = z \\ w_3 = ww' \end{cases}$

So $\psi(\frac{f}{s}.\frac{a}{t}) = \frac{a}{t}.\phi(\frac{f}{s})$.

Then ψ is well a morphism of left $(\overline{S})^{-1}B-$module.

c) Finally, let us show that ψ is bijective.
For this let us consider canonical isomorphisms of groups:

$$\begin{array}{rccl} \psi: & \overline{S}^{-1}Hom_A(M,N) & \to & Hom_A(M,S^{-1}N) \\ & \frac{f}{s} & \mapsto & f_\delta: \begin{array}{ccc} M & \to & S^{-1}N \\ m & \mapsto & \frac{f(m)}{s} \end{array} \end{array}$$

and

$$\begin{array}{rccl} \theta: & Hom_A(M,S^{-1}N) & \to & Hom_{S^{-1}A}(S^{-1}M,S^{-1}N) \\ & g & \mapsto & \tilde{g}: \begin{array}{ccc} S^{-1}M & \to & S^{-1}N \\ \frac{m}{t} & \mapsto & \frac{g(m)}{t}. \end{array} \end{array}$$

By construction, we have $\psi = \theta \circ \varphi$
φ and θ are canonical isomorphisms of groups.
So $\psi = \theta \circ \varphi$ is bijective.
Therefore, ψ is an isomorphism of left $(\overline{S})^{-1}B-$module.

□

Corollary 1.11. *Let A be a ring (not necessarily commutative), S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions in A, M an $(A-A)-$bimodule. Then for every left $A-$module N, we have the following isomorphism of left $S^{-1}A-$modules:*

$$\begin{array}{rccl} \psi: & S^{-1}Hom_A(M,N) & \to & Hom_{S^{-1}A}(S^{-1}M,S^{-1}N)) \\ & \frac{f}{s} & \mapsto & \frac{1}{s}S^{-1}(f) \end{array}.$$

Proof. It suffices to take $B = A$ in **Theorem 1.10**. Indeed, we can localize A in the right because S satisfies the right Ore's conditions in A. □

Proposition 1.12. *Let B be a duo-ring, A a subring of B, ${}_AM$ a left $A-$module, ${}_BN_A$ $(B-A)-$bimodule and ${}_BQ_A$ a $(B-A)-$bimodule, S a multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S satisfying the right Ore's conditions in B. We have the following isomorphism of right $S^{-1}A-$modules:*

$$Hom_{S^{-1}A}(S^{-1}N \otimes S^{-1}M, S^{-1}Q) \cong Hom_{S^{-1}A}(S^{-1}M, Hom_{S^{-1}A}(S^{-1}N, S^{-1}Q).$$

Proof. It results from the adjoint isomorphism theorem (see [9] and the fact that the functor $S^{-1}()$ commutes with the functor $Hom(Q,-)$ for any $(B-A)-$module ${}_BQ_A$ according to **Theorem 1.10**. □

2. Functor Tor and functor $S^{-1}()$

In this section, B denotes an unit duo-ring, A a subring of B, S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S.
We show that the functor $S^{-1}()$ commutes with the functor Tor.

Theorem 2.1. *Let M be a left $A-$module (respectively a $(B-A)-$bimodule), N a $(B-A)-$bimodule (respectively a left $A-$module). Then for every integer $n \geq 0$, we have the following isomorphism of left $(\overline{S})^{-1}B-$module :*

$$(\overline{S})^{-1}Tor_n^A(M,N) \cong Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Proof. It suffices to prove this theorem in the case where M is a left $A-$module and N a $(B-A)-$bimodule. It's clear that $Tor_n^A(M,N)$ is a left $B-$module, so $(\overline{S})^{-1}Tor_n^A(M,N)$ is a left $(\overline{S})^{-1}B-$module. Also $S^{-1}M$ is a left $S^{-1}A$- module; $S^{-1}N$ is a $(S^{-1}B - S^{-1}A)$- bimodule, so $Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N)$ is a left $(\overline{S})^{-1}B-$module.

a) For $n = 0$ we have

$$Tor_0^A(M,N) \cong M \otimes_A N \quad \text{and}$$
$$Tor_0^{S^{-1}A}(S^{-1}M, S^{-1}N) \cong S^{-1}M \otimes_A S^{-1}N \quad (\text{see } [9]).$$

Otherwise, $S^{-1}()$ is an exact and additive functor so according to [2], it commutes with tensor product functor $M \otimes_A -$ and then we have: $(\overline{S})^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_A S^{-1}N$, hence the result.

b) Let now P_N be a projective resolution of N
As the functor S^{-1} keep projectivity so $S^{-1}(P_N)$ is a projective resolution of $S^{-1}N$.
According to **Theorem 1.10**, Proving the existence of isomorphism ψ, we deduce the isomorphism of the complexes

$$(\overline{S})^{-1}(M \otimes {}_AP_N) \cong S^{-1}M \otimes {}_{S^{-1}A}S^{-1}(P_N).$$

Hence their homology groups are isomorphic and as the functor $S^{-1}()$ is exact and by definition of the functor Tor, we have

$$H_n(S^{-1}(M \otimes {}_AP_N) \cong S^{-1}H_n(M \otimes {}_AP_N) \cong (\overline{S})^{-1}Tor_n^A(M,N)$$

As well as $S^{-1}(P_N)$ is a projective resolution of $S^{-1}N$, so

$$H_n(\overline{S})^{-1}M \otimes {}_{S^{-1}A}S^{-1}(P_N)) \cong Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N).$$

□

Corollary 2.2. *Let A be a duo-ring, P a prime ideal of A, S the set of regular elements of $A - P$, M a left $A-$module and N an $(A-A)-$bimodule. Denote respectively by A_P the ring of fractions $S^{-1}A$ of A in S, M_P the module of fractions $S^{-1}M$ and N_P the module of fractions $S^{-1}N$*
So we have the following isomorphism of left A_P-modules :

$$Tor_n^A(M,N)_P \cong Tor_n^{A_P}(M_P, N_P).$$

Proof. According to [7], the ring of fractions A_P and the modules of fractions M_P and N_P exist. It is then sufficient to apply the **Theorem 2.1**. □

Theorem 2.3. *Let B be a duo-ring, A a subring of B, ${}_BM_A$ a $(B-A)-$bimodule, S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S, then the correspondence $Tor_n^{S^{-1}A}(S^{-1}M, -) : S^{-1}A - Mod \to (\overline{S})^{-1}B - Mod$ defined by:*

a) *Action on objects: for any left $S^{-1}A-$module $S^{-1}N$, associates the left $(\overline{S})^{-1}B-$module $Tor_{S^{-1}A}^n(S^{-1}M, S^{-1}N)$*

b) *Action on morphisms: for any morphism of left $S^{-1}A-$modules $f : S^{-1}N \to S^{-1}Q$, associates the morphism of left $(\overline{S})^{-1}B-$modules $Tor^n_{S^{-1}A}(S^{-1}M, -)(f) : (\overline{S})^{-1}f(N) \to (\overline{S})^{-1}f(Q)$ is a functor.*

Proof. It's clear that with the conditions of this theorem, $Tor^n_{S^{-1}A}(S^{-1}M, S^{-1}N)$ is a left $(\overline{S})^{-1}B-$module.
Furthermore let $S^{-1}N$ be a left $S^{-1}A-$module, for any morphism $f : S^{-1}N \to S^{-1}Q$, we define

$$Tor^n_{S^{-1}A}(S^{-1}M, -)(f) = Tor^n_{S^{-1}A}(S^{-1}M, f) = H_n(1_{S^{-1}M} \otimes_{S^{-1}A} f)$$

where $H_n(1_{S^{-1}M} \otimes_{S^{-1}A} f)$ is a projective chain of f. Then $Tor^n_{S^{-1}A}(S^{-1}M, f)$ is a morphism of left $S^{-1}A-$modules(see [2] and [9]). □

3. Functor Ext and functor $S^{-1}()$

In this section, B denotes an unit duo-ring, A a subring of B and S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S.
We show that the functor $S^{-1}()$ commutes with the Ext functor.

Theorem 3.1. *let M be a left $A-$module(respectively a $(A-B)-$bimodule), N a $(A-B)-$bimodule (respectively a left $A-$module).*
Then for every integer $n \geq 0$, we have the following isomorphism of right(respectively a left) $(\overline{S})^{-1}B-$modules $(\overline{S})^{-1}Ext^n_A(M, N) \cong Ext^n_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$.

Proof. It suffices to prove this theorem in the case where M is a left $A-$module and N a $(A-B)-$bimodule.
It's clear that $Ext^n_A(M, N)$ is a right $B-$module, so $(\overline{S})^{-1}Ext^n_A(M, N)$ is a right $(\overline{S})^{-1}B-$module. Also $S^{-1}M$ is a left $S^{-1}A-$ module; $S^{-1}N$ is a $(S^{-1}A - S^{-1}B)-$bimodule, so $Ext^n_{S^{-1}A}(S^{-1}M, S^{-1}N)$ is a right $(\overline{S})^{-1}B-$module.
Now let P_M be a projective resolution of M.
According to **Theorem 1.10**, there is an isomorphism
$\psi : (\overline{S})^{-1}Hom_A(M, N) \to Hom_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$.
We deduce the isomorphism of the complexes:
$(\overline{S})^{-1}\left(Hom_A(P_M, N)\right) \cong Hom_{S^{-1}A}\left(S^{-1}(P_M), S^{-1}N\right)$
By applying the functor homology H_n we have:
$H_n\left((\overline{S})^{-1}\left(Hom_A(P_M, N)\right)\right) \cong (\overline{S})^{-1}H_n\left(Hom_A(P_M, N)\right) \cong (\overline{S})^{-1}Ext^n_A(M, N)$
since the functor $S^{-1}()$ is exact, therefore we have:

$$H_n\left(Hom_{S^{-1}A}\left(S^{-1}P_M, S^{-1}N\right)\right) = Ext^n_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$$

because $S^{-1}(P_M)$ is a projective resolution. □

Corollary 3.2. *Let A be a duo-ring, P a prime ideal of A, S the set of regular elements of $A - P$, M a left $A-$module and N an $(A-A)-$bimodule. Denote respectively by A_P the ring of fractions $S^{-1}A$ of A in S, M_P the module of fractions $S^{-1}M$ and N_P the module of fractions $S^{-1}N$*
So we have the following isomorphism of right A_P-modules :

$$Ext^n_A(M, N)_P \cong Ext^n_{A_P}(M_P, N_P).$$

Proof. It results from [5] and from **Theorem 3.1**. □

Theorem 3.3. *Let B be a duo-ring, A a subring of B, ${}_BM_A$ a $(B-A)-$bimodule, S a non-empty multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S, then the correspondence $Ext^n_{(\overline{S})^{-1}B}(S^{-1}M,-):(\overline{S})^{-1}B-Mod\to S^{-1}A-Mod$ defined by:*

a) *Action on objects: for any left $(\overline{S})^{-1}B-$module $S^{-1}N$, associates the left $(S^{-1}A-$module $Ext^n_{(\overline{S})^{-1}B}(S^{-1}M,S^{-1}N)$*

b) *Action on morphisms: for any morphism of left $(\overline{S})^{-1}B-$modules $f:S^{-1}N\to S^{-1}Q$, associates the morphism of left $S^{-1}A-$modules*

$$Ext^n_{(\overline{S})^{-1}B}(S^{-1}M,-)(f):S^{-1}f(N)\to S^{-1}f(Q),\ \textit{is a functor.}$$

Proof. It's clear that with the conditions of this theorem, $Ext^n_{(\overline{S})^{-1}B}(S^{-1}M,S^{-1}N)$ is a left $S^{-1}A-$module.
Furthermore let $S^{-1}N$ be a left $(\overline{S})^{-1}B-$module, for any morphism $f:S^{-1}N\to S^{-1}Q$, we define

$$Ext_{(\overline{S})^{-1}B}(S^{-1}M,-)(f)=Ext_{(\overline{S})^{-1}B}(S^{-1}M,f)=H_n(1_{S^{-1}M}\otimes_{(\overline{S})^{-1}B}f)$$

where $H_n(1_{S^{-1}M}\otimes_{(\overline{S})^{-1}B}f)$ is an injective chain of f. Then $Ext_{(\overline{S})^{-1}B}(S^{-1}M,f)$ is a morphism of left $(\overline{S})^{-1}B-$modules(see [2] and [8]). □

4. Adjoint functors of Functors $S^{-1}()$, Ext and Tor

In this section, unless otherwise stated, B denotes a ring (not necessarily commutative), A a subring of B, M a left $A-$module, S a multiplicatively closed subset of the ring A satisfying the left Ore's conditions, $S^{-1}A$ the ring of fractions of A on S.
We know that (see [10]) the functor $Ext^0_A(M,-)$ Is equivalent to the functor $Hom_A(M-)$, $Ext^0_A(-,M)$ Is equivalent to the functor $Hom_A(-,M)$ and $Tor^A_0(M,-)$ Is equivalent to the functor $M\otimes_A-$ which is isomorphic to the functor $S^{-1}()$.
So $Ext^0_{S^{-1}A}(S^{-1}A,-)$ is adjoint to $S^{-1}()$ and $Tor_0^{S^{-1}A}(S^{-1}A,-)$ is isomorphic to $S^{-1}()$.
The aim of this section is to establish a generalization of these properties between the functor $S^{-1}()$ and functors $Ext^n_{S^{-1}B}(S^{-1}M,-)$ and $Tor_n^{S^{-1}A}(S^{-1}M,-)$ respectively for any integer n.

Definition 4.1. ***Adjoint Functors***

Let Λ and Γ be two categories, $F:\Lambda\to\Gamma$ and $G:\Gamma\to\Lambda$ two functors. We say that the pair (F,G) is adjoint if for all $A\in Ob(\Lambda)$ and for all $B\in Ob(\Gamma)$, there is an isomorphism $r_{A,B}:Hom_\Lambda(A,G(B))\to Hom_\Gamma(F(A),B)$ such as :

a) *for all $f\in Hom_\Lambda(A,A')$, the following diagram is commutative:*

$$\begin{array}{ccc} Hom_\Lambda(A,G(B)) & \overset{f_*=Hom(f,G(B))}{\longrightarrow} & Hom_\Lambda(A',G(B)) \\ \downarrow r_{A,B} & & \downarrow r_{A',B} \\ Hom_\Gamma(F(A),B) & \overset{F(f)_*=Hom(F(f),B)}{\longrightarrow} & Hom_\Gamma(F(A'),B) \end{array}$$

b) *for all $g\in Hom_\Gamma(B,B')$, the following diagram is commutative:*

$$\begin{array}{ccc} Hom_\Lambda(A,G(B)) & \overset{(G(g))^*=Hom(A,G(g))}{\longrightarrow} & Hom_\Lambda(A,G(B')) \\ \downarrow r_{A,B} & & \downarrow r_{A',B} \\ Hom_\Gamma(F(A),B) & \overset{g^*=Hom(F(A),g)}{\longrightarrow} & Hom_\Gamma(F(A),B') \end{array}$$

Proposition 4.2. *Let A and B be two rings, ${}_AM_B$ an (A,B) bimodule, ${}_BN_A$ an $(B,A)-$bimodule, ${}_BP$ a left $B-$module, then we have the following isomorphism of left $B-$module :*

$$Ext^n_A(M, Hom_B(N,P)) \cong Hom_B(Tor^A_n(M,N), P).$$

Proof. It's clear that with the conditions of this proposition, $Hom_B(N,P))$ is a left $A-$module, $Tor^A_n(M,N)$ is a $(B,B)-$bimodule; then $Ext^n_A(M, Hom_B(N,P))$ and $Hom_B(Tor^A_n(M,N),P)$ are left $B-$modules.
For the rest of the proof, see [3]. □

Proposition 4.3. *Let B be a noetherian duo-ring, A a subring of B, ${}_AM_B$ an (A,B) bimodule, ${}_BN_A$ a $(B-A)-$bimodule, ${}_BP$ a left injective $B-$module, S a multiplicatively closed subset of the ring A satisfying the left and right Ore's conditions, $\overline{S}$ the set of regular elements of S satisfying the left and right Ore's conditions in B, then we have the following isomorphism of left $(\overline{S})^{-1}B-$modules:*

$$Ext^n_{S^{-1}A}(S^{-1}M, Hom_{(\overline{S})^{-1}B}(S^{-1}N, S^{-1}P)) \cong Hom_{(\overline{S})^{-1}B}(Tor^A_n(S^{-1}M, S^{-1}N), S^{-1}P).$$

Proof. As B is noetherian, the functor $S^{-1}()$ preserves injectivity (see [2]), then $S^{-1}P$ is a left injective $S^{-1}A$ module, This result is obtained by applying **Theorem 1.10** and **Proposition 4.2**. □

Proposition 4.4. *Let A and B be two rings, ${}_BM_A$ a (B,A) bimodule, ${}_BN_A$ an $(B,A)-$bimodule, ${}_BP$ a left $B-$module, then we have the following isomorphism of left $B-$module :*

$$Tor^B_n(Hom_A(N,M), P) \cong Hom_B(Ext^n_A(M,N), P).$$

Proof. With these conditions, $Hom_A(N,M))$ and $Ext^n_A(M,N)$ are $(B,B)-$bimodules; then $[Tor^B_n(Hom_A(N,M),P)$ and $Hom_B(Ext^n_A(M,N),P)$ are left $B-$modules.
For the rest of the proof, see [3]. □

Theorem 4.5. *Let B be a noetherian duo-ring, A a subring of B, ${}_BM_A$ a (B,A) bimodule, ${}_BN_A$ an $(B,A)-$bimodule, ${}_BP$ a left injective $B-$module, S a multiplicatively closed subset of the ring A satisfying the left Ore's conditions, $\overline{S}$ the set of regular elements of S.*
Then we have the following isomorphism of left $(\overline{S})^{-1}B-$modules:

$$Tor^{(\overline{S})^{-1}B}_n(Hom_{S^{-1}A}(S^{-1}N, S^{-1}M), S^{-1}P) \cong Hom_{(\overline{S})^{-1}B}(Ext^n_{S^{-1}A}(S^{-1}M, S^{-1}N), S^{-1}P).$$

Proof. The proof is similar to that of **Proposition 4.3**. □

Theorem 4.6. *Let B be a noetherian duo-ring, A a subring of B, ${}_BM_A$ a $(B-A)-$bimodule, ${}_BN_A$ a $(B-A)-$bimodule, ${}_BP$ a left injective $B-$module, S a multiplicatively closed subset of the ring A satisfying the left Ore's conditions, $\overline{S}$ the set of regular elements of S, $S^{-1}A$ the ring of fractions of A on S. Then covariant functors:* $Ext^n_{(\overline{S})^{-1}B}(S^{-1}M, -) : (\overline{S})^{-1}B - Mod \to S^{-1}A - Mod$

$$\text{and} \quad Tor^{S^{-1}A}_n(S^{-1}M, -) : S^{-1}A - Mod \to (\overline{S})^{-1}B - Mod \quad \text{are adjoint.}$$

Proof. To prove this theorem we assume that N is projective and P is injective, consequently the functors $Hom_A(-,P)$ and $Hom_B(N,-)$ are exact.

(1) According to the adjoint isomorphism theorem, we have $Hom_A(N, Hom_B(M,P)) \cong Hom_B(M \otimes_A N, P)$, hence

$$(\overline{S})^{-1}Hom_A(N, Hom_B(M,P)) \cong (\overline{S})^{-1}Hom_B(M \otimes_A N, P)$$

and so

$$Hom_{S^{-1}A}(S^{-1}N, Hom_{(\overline{S})^{-1}B}(S^{-1}M, S^{-1}P)) \cong Hom_{(\overline{S})^{-1}B}(S^{-1}M \otimes_{S^{-1}A} S^{-1}N, S^{-1}P)$$

applying **Theorem 1.10**.
$S^{-1}N$ is projective module and $S^{-1}P$ is injective, therefore
functors $Hom_{S^{-1}A}(S^{-1}N, -)$ and $Hom_{(\overline{S})^{-1}B}(-, S^{-1}P)$ are exact.
Therefore, we have:

$H_n[Hom_{S^{-1}A}(S^{-1}N, Hom_{(\overline{S})^{-1}B}(S^{-1}M, S^{-1}P))]$ is isomorphic to

$$H_n[Hom_{(\overline{S})^{-1}B}(S^{-1}M \otimes_A S^{-1}N, S^{-1}P)]$$

which gives

$Hom_{S^{-1}A}[H_n(S^{-1}N, Hom_{(\overline{S})^{-1}B}(S^{-1}M, S^{-1}P))]$ is isomorphic to

$$Hom_{(\overline{S})^{-1}B}[H_n(S^{-1}M \otimes_{S^{-1}A} S^{-1}N, S^{-1}P)].$$

Let X_N be a projective resolution of $S^{-1}N$ and Y_P an injective resolution of $S^{-1}P$. Then we have

$$Hom_A[H_n(X_N, Hom_B(S^{-1}M, S^{-1}P))] \cong Hom_B[H_n((\overline{S})^{-1}M \otimes_A S^{-1}N, Y_P)]$$

which gives :

$Hom_{S^{-1}A}[S^{-1}N, Ext^n_{(\overline{S})^{-1}B}(S^{-1}M, S^{-1}P)]$ is isomorphic to

$$Hom_{(\overline{S})^{-1}B}[Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}N), S^{-1}P].$$

(2) Let us now prove by induction on integer n than the following two diagrams, where $F = (Tor_n^{S^{-1}A}(S^{-1}M, -))$ and $G = Ext^n_{(\overline{S})^{-1}B}(S^{-1}M, -)$ are commutative:

Let $f : S^{-1}N \longrightarrow S^{-1}N'$ and $g : S^{-1}P \longrightarrow S^{-1}P'$

$$\begin{array}{ccc} Hom_{S^{-1}A}(S^{-1}N, G(S^{-1}P)) & \xrightarrow{f_* = Hom(f, G(S^{-1}P))} & Hom_{S^{-1}A}(S^{-1}N', G(S^{-1}P)) \\ \downarrow r_{S^{-1}N, S^{-1}P} & & \downarrow r_{S^{-1}N', S^{-1}P} \\ Hom_{(\overline{S})^{-1}B}(F(S^{-1}N), S^{-1}P) & \xrightarrow{(Ff)_* = Hom(F(f), S^{-1}P)} & Hom_{(\overline{S})^{-1}B}(F(S^{-1}N'), S^{-1}P). \end{array}$$

$$\begin{array}{ccc} Hom_{S^{-1}A}(S^{-1}N, G(S^{-1}P)) & \xrightarrow{G(g)^* = Hom(S^{-1}N, G(g))} & Hom_{S^{-1}A}(S^{-1}N, G(S^{-1}P')) \\ \downarrow r_{S^{-1}N, (\overline{S})^{-1}B} & & \downarrow r_{S^{-1}N', S^{-1}P} \\ Hom_{(\overline{S})^{-1}B}(F(S^{-1}N), S^{-1}P) & \xrightarrow{g^* = Hom(F(S^{-1}N), g)} & Hom_{(\overline{S})^{-1}B}(F(S^{-1}N), S^{-1}P'). \end{array}$$

a) For $n = 0$, we have $Ext^0_{(\overline{S})^{-1}B}(S^{-1}M, -) \cong Hom_{S^{-1}B}(S^{-1}M, -)$ and $Tor_0^{(\overline{S})^{-1}B}(S^{-1}M, -) \cong S^{-1} \otimes_{S^{-1}A} -$ now (see [6] and [8], the functors $Hom_{(\overline{S})^{-1}B}(S^{-1}M, -) : (\overline{S})^{-1}B - Mod \to S^{-1}A - Mod$ and $M \otimes_{(\overline{S})^{-1}B} - : S^{-1}A - Mod \to (\overline{S})^{-1}B - Mod$ are adjoint. From this, results that diagrams (1) and (2) are commutative in taking $F = M \otimes_{S^{-1}A} -$ and $G = Hom_{(\overline{S})^{-1}B}(S^{-1}M, -)$

b) Suppose that for any integer $k \leqslant n$, diagrams (1) and (2) are commutative where $F = Tor_k^{S^{-1}A}(S^{-1}M, -)$ and $G = Ext^k_{(\overline{S})^{-1}B}(S^{-1}M, -)$.
We show that (1) and (2) are commutative by taking

$$F = Tor_{n+1}^{S^{-1}A}(S^{-1}M, -) \text{ and } G = Ext^{n+1}_{S^{-1}B}(S^{-1}M, -).$$

Let $P_{S^{-1}M} : \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} S^{-1}N \to 0$ a projective resolution of $S^{-1}M$ of n^{th} kernel $K_n = Ker(d_n)$.
Otherwise, we have (see [2]) :

$$Ext^{n+1}_{S^{-1}A}(S^{-1}M, S^{-1}P) \cong Ext^1_{S^{-1}A}(K_{n-1}, S^{-1}P) \text{ and}$$

$$Tor_{n+1}^{S^{-1}A}(S^{-1}M, S^{-1}N) \cong Tor_1^{S^{-1}A}(K_{n-1}, S^{-1}N)$$

Hence, it is enough to apply the hypothesis of recurrence to the functors $Ext^1_{S^{-1}A}(K_{n-1}, -)$ and $Tor_1^{S^{-1}A}(K_{n-1}, -)$.

c) We conclude that for every integer n, the functors $Ext^n_{S^{-1}B}(S^{-1}M, -)$ and $Tor_n^{S^{-1}A}(S^{-1}M, -)$ are adjoint.

The proof is similar to that of **Proposition 4.3**. □

Corollary 4.7. *Let A be a noetherian ring, S a multiplicatively closed subset of the ring A satisfying the left Ore's conditions, $S^{-1}A$ The ring of fractions of A on S. Then functors $Ext^n_{S^{-1}A}(S^{-1}A, -)$ and $Tor_n^{S^{-1}A}(S^{-1}A, -)$ are adjoint.*

Proof. It results from **Theorem 4.6** and in the fact that the ring A is an $(A - A)-$bimodule. □

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